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ILL-POSEDNESS AND REGULARIZATION OF AN INVERSE PROBLEM OF VOLATILITY IDENTIFICATION

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INTRODUCTION

Inverse problems in option pricing and corresponding regularization approaches including convergence rates results have found increasing interest in the past ten years. Substantial contributions to that topic have been published, for example, by Bouchouev & Isakov 1997 and 1999, Lagnado & Osher 1997, Jackson, Süli & Howison 1998, Crépey 2003, and Egger & Engl 2005.

In the first part of this presentation, we consider a specific nonlinear inverse problem of this scenery, the problem of calibrating a purely time-dependent volatility function $\sigma(t)$ of a price process $X(t)$ for an asset satisfying the stochastic differential equation

$$\frac{dX(t)}{X(t)} = \mu\, dt + \sigma(t)\, dW(t) \quad (t \geq 0,\ X^* = X(0) > 0)$$

with standard Wiener process $W(t)$ and drift μ. The calibration is to be done for a fixed time interval $[0,T]$ based on maturity-dependent prices $u(t)$ $(0 \leq t \leq T)$ of European vanilla call options with fixed strike $K^* > 0$, see [1].

Inverse problems in option pricing are frequently regarded as simple and resolved if a formula of Black-Scholes-type defines the forward operator. However, precisely because the structure of such problems is straightforward, they may serve as benchmark problems for studying the nature of ill-posedness occurring in the context of volatility calibration. Moreover, by following the decoupling approach suggested in [2] variants of this benchmark problem also occur as serious subproblems for the recovery of local volatility surfaces. Such surfaces are of considerable practical importance in finance.

ILL-POSEDNESS OF THE INVERSE PROBLEM AND THE PROBLEM DECOMPOSITION

Setting $a(t) := \sigma^2(t)$ and $[J\,a](t) := \int_0^t a(\tau)\, d\tau$ $(0 \leq t \leq T)$ the inverse problem can be formulated as an operator equation

$$F(a) = u, \tag{1}$$

where the nonlinear forward operator $F : D(F) \subset B_1 \to B_2$ with half-space domain $D(F)$ maps between Banach or Hilbert spaces B_1 and B_2 of continuous or integrable functions over the interval $[0,T]$. The associated forward operator $F : a \mapsto u$ has the form

$$[F(a)](t) := U_{BS}(X^*, K^*, r^*, t, [Ja](t)) \quad (0 \leq t \leq T)$$

with Black-Scholes function U_{BS} having the current asset price X^*, the strike price K^* and the risk-free interest rate r^* as parameters. In other words, we have a composition $F = N \circ J$ with the nonlinear Nemytskii operator $[N(S)](t) := k(t, S(t))$ $(0 \leq t \leq T)$ for

$$k(t,v) = U_{BS}(X^*, K^*, r^*, t, v) \quad ((t,v) \in [0,T] \times [0,\infty)).$$

Consequently, the operator equation (1) can be decomposed into a nonlinear outer equation

$$N(S) = u, \tag{2}$$

and a linear inner equation.

$$J\,a = S. \tag{3}$$

Since J is compact for spaces B_1 and B_2 of continuous or integrable functions, it is well-known that (3) is moderately ill-posed. We show the continuity of N, which implies the compactness of F. Hence the total inverse problem (1) is ill-posed and its stable approximate solution requires the use of a regularization method.

ILL-CONDITIONING OF THE NONLINEAR OUTER PROBLEM

We emphasize that properties of the outer nonlinear problem (2) depend on the choice of the function spaces B_1 and B_2, respectively their specific norms under consideration. So it can be shown that (2) is ill-posed in $B_1 = B_2 = L^2(0,T)$, but well-posed in $B_1 = B_2 = C[0,T]$. In the latter case, however, at least a significant ill-conditioning effect can be observed by numerical case studies.

CONVERGENCE RATES FOR THE TIKHONOV REGULARIZATION

Using a Hilbert space setting we analyze the convergence of the Tikhonov regularization for the inverse problem (1). The nature of ill-posedness of the nonlinear composition problem is considered by means of the character of regularized solutions using Tikhonov's method. In particular, there is given a detailed analysis of convergence rates including the interpretation of corresponding source conditions, which are related to multiplication operators, see [3]. This analysis only applies for options in the money and out of the money, i.e., for $X^* \neq K^*$.

THE SINGULAR CASE OF AT-THE-MONEY OPTIONS

The second part of this presentation is concerned with the singular case of at-the-money options with $X^* = K^*$, which plays an interesting role for the benchmark problem. For that case the Fréchet derivative $F'(a)$ of the forward operator degenerates. Consequently, the classical analysis of convergence rates originally established by Engl, Kunisch & Neubauer 1989 cannot be applied directly. The talk presents an alternative approach based on Bregman distances bridging the gap between this singular case and the case of in-the-money and out-of-the-money options with respect to the benchmark problem, see [4].

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